Correlation-Function Inequality Obtained by Yeh

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An inequality on correlation functions in Heisenberg ferromagnets derived by Yeh is slightly improved, but shown to be irrelevant to the problem of phase transitions.

ONE motivation for the study of various correlation-function inequalities in Ising ferromagnets[1] has been their applicability to the problem of phase transitions. Attempts to extend these results to non-Ising systems[2] have not been very successful to date. Recently, Yeh[3] has published an upper bound for pair correlations in Heisenberg ferromagnets with the suggestion that it might prove useful in discussions of phase transitions. However, even an improved version (below) of Yeh's inequality turns out to be rather trivial in a large system, and hence of no value in the phase-transition problem.

Consider a system of spins $\boldsymbol{\sigma}_i$, $i=1, 2, \ldots N$, with a Hamiltonian H chosen so that it has no negative eigenvalues. This can always be done by adding an (N-dependent) constant to the Hamiltonian of interest. The inequalities[3]

$$e^{-\beta\langle\psi|H|\psi\rangle}\leq\langle\psi|e^{-\beta H}|\psi\rangle\leq 1 \tag{1}$$

hold for any normalized wave function ψ. We shall, in particular, choose ψ's which are products of the eigenstates of σ_{iz}, $i=1, \ldots N$. The pair correlation function is given by

$$\langle\sigma_{kz}\sigma_{lz}\rangle=\Big(\sum_P\langle\psi|e^{-\beta H}|\psi\rangle-\sum_A\langle\psi|e^{-\beta H}|\psi\rangle\Big)\Big/\Big(\sum_P\langle\psi|e^{-\beta H}|\psi\rangle+\sum_A\langle\psi|e^{-\beta H}|\psi\rangle\Big), \tag{2}$$

where $\sum_P(\sum_A)$ is the sum over states in which $\sigma_{kz}\sigma_{lz}=+1(-1)$, that is, states for which these spins are parallel (antiparallel).

Upon noting that $(w-1)/(w+1)$ is monotone increasing for $w>-1$, we see that the bounds

$$\sum_P\langle\psi|e^{-\beta H}|\psi\rangle\leq 2^{N-1}, \tag{3}$$

$$\sum_A\langle\psi|e^{-\beta H}|\psi\rangle\geq\sum_A e^{-\beta\langle\psi|H|\psi\rangle} \tag{4}$$

may be combined with (2) to yield

$$\langle\sigma_{kz}\sigma_{lz}\rangle\leq(R-1)/(R+1), \tag{5}$$

where

$$R=2^{N-1}/\sum_A e^{-\beta\langle\psi|H|\psi\rangle}.$$

Yeh's somewhat weaker bound is obtained by replacing $(R+1)$ in (5) by 2 (note that $R\geq 1$).

To see what happens to R for a large system, consider $r\leq R$ defined by

$$r=2^{N-1}/\sum_A\langle\psi|e^{-\beta H}|\psi\rangle=e^{G(0)-G(\beta)},$$

where $G(\beta)$ is $-\beta$ times the free energy of the system corresponding to a "boundary" condition $\sigma_{kz}\sigma_{lz}=-1$. It is at least in general true for systems of the usual sort considered in phase transitions that the free energy is an extensive quantity. In addition, the aforementioned boundary condition has a negligible effect on G for large N, as is easily shown using procedures developed in an earlier paper.[4] [For readers unfamiliar with the type of argument involved, we consider the slightly simpler case of a boundary condition $\sigma_{kz}=+1$. The partition function with this constraint differs by at most a factor of $2e^{\beta\epsilon}$ from the partition function in which all exchange interactions involving the kth spin have been set equal to zero, which in turn differs by at most the same factor from the partition function in which all the exchange interactions have their original values, but no constraint is placed on σ_{kz}. The constant ϵ, defined in Eq. (45) of Ref. 4, is independent of N. By an obvious extension of this argument, one shows that the condition $\sigma_{kz}\sigma_{lz}=-1$ adds to G, which is of order N, a term which is (at most) of order 1.] Consequently, $G(0)-G(\beta)$ is (approximately) proportional to N; thus r (and *a fortiori* R) increases extremely rapidly, and neither Yeh's bound (increasing to infinity) nor the bound in (5) (rapidly approaching 1) is of any use when $N\to\infty$, the interesting limit for the problem of phase transitions.

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[1] J. Ginibre, Phys. Rev. Letters **15**, 828 (1969), and references therein.

[2] C. A. Hurst and S. Sherman, Phys. Rev. Letters **22**, 1357 (1969).

[3] R. H. T. Yeh, Phys. Rev. Letters **23**, 1241 (1969).

[4] R. B. Griffiths, J. Math. Phys. **5**, 1215 (1964). The reader is referred to this paper for a more precise statement of conditions sufficient to guarantee that G is extensive.